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A NON-IMMERSION THEOREM FOR REAL PROJECTIVE SPACE

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§1. INTRODUCTION

LET P_n denote real projective n-space, with its usual differential structure. The purpose of this note is to prove

THEOREM (1.1). *P_n cannot be immersed in $(2^{r+1}-1)$-space where*

$$n = 2^r + r + 2 \qquad r \not\equiv 1 \mod 4$$
$$n = 2^r + r + 3 \qquad r \equiv 1 \mod 4.$$

The method of Stiefel–Whitney classes shows that if $n = 2^r$ then P_n, and consequently P_m for $m > n$, cannot be immersed in $(2^{r+1}-2)$-space. In [1] Atiyah has proved, by the method of exterior powers, that P_n cannot be immersed in $(2n - p)$-space where p is approximately $n/2$. Theorem (1.1) first gives a new result when $n = 22$. Levine [3] and Mahowald [4] proved the analogous non-embedding theorem; P_n cannot be embedded in 2^{r+1}-space when $n = 2^r + 1$. Our theorem may be true when $n = 2^r + 2$ (if $n = 2^r + 1$ the immersion is possible (5.3) of [6], or see (5.7) of [5]). Improvements on the Stiefel–Whitney class result for complex and quaternion projective space can be obtained from (1.1) by using the fibrations of projective spaces (see (5.2) of [6]). However better results are available in this case [7].

Our methods rely heavily on the work of James, and I am indebted to him for sending me a preprint of [2].

§2. IMMERSIONS AND AXIAL MAPS

Suppose that P_q has its standard cell structure with base point e. When $r > q$ regard P_r, in the usual way, as a subspace of P_q. An *axial map* of $P_s \times P_t$ into P_q is a map

$$f : P_s \times P_t \to P_q$$

such that if $x \in P_s$ and $y \in P_t$ then

$$f(x, e) = x, \; f(e, y) = y.$$

In this section we shall prove

THEOREM (2.1). *If P_n is immersible in $(n + k)$-space then there exists an axial map of $P_n \times P_n$ into P_{n+k}. The converse is true if $n < 2k$.*

Let $K\tilde{O}(P_n)$ denote the reduced Grothendieck group of classes of real vector bundles over P_n. A generator of this group is the class x of the non-trivial line bundle ξ over P_n. The group is cyclic of order a power of 2 and multiplicative structure is given by $x^2 = -2x$. Let $g(rx)$ denote the *geometrical dimension*, as defined in §1 of [1], of the element $rx \in K\tilde{O}(P_n)$. The proof of (2.1) rests on the following lemma and the work of James [2].

LEMMA (2.2). *$g(rx) \leqslant k$ if and only if $g((k-r)x) \leqslant k$.*

Proof. Suppose $g(rx) \leqslant k$, then there exists a real vector bundle η such that $\theta(\eta) = rx + k$, θ being the homomorphism defined in [1]. Now $\theta(\eta \otimes \xi) = (rx + k)(x + 1) = (k - r)x + k$. Conversely if there exists a bundle ζ such that $\theta(\zeta) = (k - r)x + k$ then $\theta(\zeta \otimes \xi) = rx + k$.

The class of the tangent bundle of P_n is $(n + 1)x$, hence from (2.1) of [6], or see §3 of [1], P_n is immersible in $(n + k)$-space if and only if $g(-(n + 1)x) \leqslant k$. Applying the lemma, with $r = -(n + 1)$, to this result we have

THEOREM (2.3). *P_n is immersible in $(n + k)$-space if and only if $g((n + k + 1)x) \leqslant k$.*

Recall from [2] that a *t-field on P_s of tangents to P_q* is a continuous function which assigns to each point of P_s a set of t linearly independent tangent vectors to P_n at that point.

LEMMA (2.4). *Suppose $s < q$, then P_s admits a t-field of tangents to P_q if and only if $g((q + 1)x) \leqslant q - t$ on P_s.*

Proof. Suppose $g((q + 1)x) \leqslant q - t$ on P_s. Then there exists a vector bundle η such that $\theta(\eta) = (q + 1)x + q - t$, and since $q > s$ the bundle sum of η with a trivial vector bundle of dimension t is isomorphic with the restriction of the tangent bundle of P_q to P_s. Hence P_s admits a t-field of tangents to P_q. The converse follows directly from the definition of geometrical dimension.

Combining (2.4) with Theorem (4.1) of [2] we have

THEOREM (2.5). *There exists an axial map of $P_s \times P_t$ into P_q if $g((q + 1)x) \leqslant q - t$ on P_s. The converse is true if $s < 2(q - t)$.*

Setting $s = t = n$ and $q = n + k$ Theorem (2.1) now follows from (2.3) and (2.5).

§3. PROOF OF THEOREM (1.1)

Let $\phi(k, n)$ denote the number of values of m in the range $k \leqslant m \leqslant n$ which are congruent to 0, 1, 2 or 4 mod 8. The following theorem is a special case of a theorem proved by James using the Adams ψ operations, (6.2) of [2].

THEOREM (3.1). *Let $C_{n+k,n}$ be odd. If there exists an axial map of $P_n \times P_n$ into P_{n+k} then*

$$n + k + 1 \text{ is a multiple of } 2^{\phi(k,n)-1} \text{ if } n \not\equiv 3 \mod 4$$

and

$$n + k + 1 \text{ is a multiple of } 2^{\phi(k,n)-2} \text{ if } n \equiv 3 \mod 4.$$

Suppose now that P_n is immersible in $(2^{r+1} - 1)$-space where n is as in (1.1) and $r > 2$ to avoid triviality. It follows from (2.1) and (3.1), since by the dyadic rule $C_{n+k,n}$ is odd, that

$$r + 1 \geqslant \phi(2^r - r - 3, 2^r + r + 2) - 1 \quad \text{if } r \not\equiv 1 \mod 4$$

and

$$r + 1 \geqslant \phi(2^r - r - 4, 2^r + r + 3) - 2 \quad \text{if } r \equiv 1 \mod 4.$$

Now $\phi(2^r - r - 3, 2^r + r + 2) = r + 3$ and $\phi(2^r - r - 4, 2^r + r + 3) = r + 4$. This provides a contradiction proving (1.1).

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